WEAK REACHABILITY AND WEAK OBSERVABILITY OF LINEAR SYSTEM OVER MAX-PLUS ALGEBRA

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Abstract. This paper discusses about the properties of linear system in max plus algebra. These properties are weak reachability and weak observability. In this case, the asticity of the system plays big role in these properties as the necessary and sufficient conditions. Furthermore, we will also discuss the duality of those properties. Finally, to make the discuss simple, we will give an example.

Keywords and Phrases: Max plus linear system, reachability, observability.

1. INTRODUCTION

The systems are changed accordingly to changes of time. But there are also systems which are changes accordingly to changes of event. Thos a kind of systems are known as event driven systems. Max plus algebra is a method which can formulate the driven event systems. These systems will be linear over max plus algebra [6].

The study of max plus algebra and its linear systems are developed widely; this study is including the theory of weak reachability and weak observability of the systems. The weak reachability means by a control system from any initial state to any other state. The systems are controlled by using the input. The difference between reachability and controllability is depending on the initial state. The reachability is the controlling the system from any initial state to any other state. But controllability is the controlling from the origin state to any other state. The concept of the reachability in the max plus algebra is not too different from the definitions of the controllability in continues system. The observability means by ability to determine the state of the system from measurement of the output. The concept about the observability in max plus algebra is also different from the observability definitions in continues systems [3].

In this paper, we will discuss about the theory of weak reachability and weak observability in the linear max plus systems. In the discussion we will use the definition of reachable and observable set. Furthermore, we also discuss about the duality among these properties and give them example.

1.1 Max Plus Algebra

In the section we explain the basic concept and notation. There are a lot of references which explain about max plus algebra, the detail information can be found in [2] and [7]. In the max plus algebra, for any \(a, b \in \mathbb{R}_\text{max} = \{-\infty\} \cup \{\mathbb{R}\}\) defined two operations, \(\oplus\) and \(\otimes\) as follows

\[
a \oplus b = \max \{a, b\} \quad \text{and} \quad a \otimes b = a + b
\]
Definition 1. For all \( x, y, z \in R_{\text{max}} \) satisfies: 1) Associative concerning \( \otimes \) and \( \oplus \). 2) Commutative concerning \( \otimes \) and \( \oplus \). 3) Distributive. 4) Zero element of \( \oplus \). 5) Unit element of \( \otimes \). 6) Multiplicative invert if \( x \neq e \) then there is \( y \) such that \( x \otimes y = e \) and \( y \) is the one and only. 7) Absorption element of \( \otimes \). 8) Idempotent in addition.

Definition 2. For \( x \in R_{\text{max}} \) and \( n \in N \) satisfies \( x^{\otimes n} = e \cdot e \cdot e \cdots e \) (n times).

Power in max plus algebra can be derived as multiplication in conventional algebra \( x^{\otimes n} = nx \), such that in generally satisfies as follows:

(i) If \( x \neq e \), then \( x^{\otimes 0} = e = 0 \) (ii) if \( \alpha \in R \), then \( x^{\otimes \alpha} = \alpha \otimes x \) (iii) if \( k > 0 \) then \( e^{\otimes k} = e \), \( e^{\otimes k} \) is undefined for \( k \leq 0 \),

1.2 Matrix over Max plus Algebra

The set of matrices size \( n \times m \) in max plus algebra denoted by \( R_{\text{max}}^{n \times m} \) with \( n, m \in N \) and \( n \) or \( m \neq 0 \). Element \( A \in R_{\text{max}}^{n \times m} \), \( i\)-th row \( j\)-th column denoted by \( a_{ij} \) or \([A]_{ij} \) for \( i = 1, 2, \ldots, n \) and \( j = 1, 2, \ldots, m \). Matrix \( A \) can be written as

\[
A = \begin{pmatrix}
a_{1,1} & a_{1,2} & \ldots & a_{1,m} \\
a_{2,1} & a_{2,2} & \ldots & a_{2,m} \\
\vdots & \vdots & & \vdots \\
a_{n,1} & a_{n,2} & \ldots & a_{n,m}
\end{pmatrix}
\]

In max plus algebra operation + and \( \times \) from vector and matrix are replaced with \( \oplus \) and \( \otimes \).

Definition 3.
1) For any \( A, B \in R_{\text{max}}^{n \times m} \) and \( \alpha \in R \) define operation \( A \oplus B \) as \([A \oplus B]_{ij} = a_{ij} \oplus b_{ij} = \max(a_{ij}, b_{ij}) \) and \( A \otimes [B]_{ij} = \alpha \otimes b_{ij} = \alpha + b_{ij} \)

2) For \( A \in R_{\text{max}}^{n \times p} \) and \( B \in R_{\text{max}}^{p \times m} \) then we define operation \( A \otimes B \) as \([A \otimes B]_{ij} = \max_{k \in p} \{a_{ik} \otimes b_{kj}\} = \max \{a_{ik} + b_{kj}\} \)

3) The transpose of matrix \( A \) denoted by \( A^T \) and defined as usual we find in conventional algebra by \([A^T]_{ij} = [A]_{ji} \).

4) Identity matrix of size \( n \times n \) in max plus is denoted by \( E_n \) and define as

\[
[E]_{ij} = \begin{cases} 
eq & \text{if } i = j \\ e & \text{if } i \neq j 
\end{cases}
\]

5) For square matrix and \( k \in N \), \( k\)-th power of \( A \) denoted by \( A^{\otimes k} \) and defined as \( A^{\otimes k} = A \otimes A \otimes A \cdots \otimes A \), for \( k = 0, A^{\otimes 0} = E_n \).

6) For matrix \( A \in R_{\text{max}}^{n \times m} \) and scalar \( \alpha \in R_{\text{max}} \), \( \alpha \otimes A \) define by \([\alpha \otimes A]_{ij} = \alpha \otimes [A]_{ij} \) for \( i, j = 1, 2, \ldots, n \) and \( j = 1, 2, \ldots, m \).
1.3 Linear Max plus System

Let the discrete event system be the event driven systems with a discrete state (as in production system, storage with finite capacity system, railway system, logistic system and so on). This state is described by the equation below:

\[ x(k + 1) = A \otimes x(k) + B \otimes u(k + 1) \quad (1) \]
\[ y(k) = C \otimes x(k) \quad (2) \]

With \( A \in \mathbb{R}^{n_{\text{max}}} \), \( B \in \mathbb{R}^{\text{max}} \), \( C \in \mathbb{R}^{\text{max}} \) and \( x \) represents the state, \( u \) represents the input and \( y \) represents the output, \( k \) is the event index which are \( k = 0, 1, 2, \ldots \). Both equation (1) and (2) are called by linear max plus system [6].

2. REACHABILITY

In this section discuss the discrete event systems which are formulated in to max plus algebra, so we get the linear one. This discussion will be done in the systems that many reaches a final condition with all of it component are greater than the final one without any input. These kinds of system later known by weakly reachable system. Using (1) in recursive fashion, the state system can be written to each event index as follows:

For \( k=0 \) then
\[ X(1) = A \otimes X(0) + B \otimes U(1) \]

For \( k=1 \) then
\[ X(2) = A \otimes X(1) + B \otimes U(2) = A \otimes X(0) + AB(U(1) + BU(2)) \]

For \( k=2 \) then
\[ X(3) = A \otimes X(2) + B \otimes U(3) = A \otimes X(0) + AB(B(U(1)) + BU(2)) \]

so, to q-step event we get:
\[ X(q) = A \otimes X(0) + \left[ B, AB, A^2B, \ldots, A^{q-1}B \right] \otimes \left[ U(q-1), U(q-2), U(Q-3), \ldots, U(0) \right] \quad (3) \]

From (3) we obtain the reachability matrix notated by \( \Gamma_q = [B, AB, A^2B, \ldots, A^{q-1}B] \). This matrix is the one which influence the reachability of the system, the input series defined by \( U_q = [U_q, U_{q-1}, \ldots, U_1] \), so the state of q-step event can be written by:
\[ X(q) = A \otimes X(0) \oplus \Gamma_q \oplus U_q \quad (4) \]

Definition 4. Reachable State. Given \( X(0) \in \mathbb{R}^n_{\text{max}} \), a state \( X \in \mathbb{R}^n \) is reachable in q-step from \( X(0) \) if there exists a control sequence \( \{U(1), U(2), \ldots, U(q)\} \in \mathbb{R}^{\text{max}} \), which achieves \( X = X(q) \).

Definition 5. Reachable Set. Let \( X(0) \in \mathbb{R}^n_{\text{max}} \), be the initial condition, the set of all of the state \( X \in \mathbb{R}^n \) that can be reached at q-step event (with q should be positive integer) is defined as follows:
\[ \Omega_{q,X(0)} = \{ X \in \mathbb{R}^n : X = A^q \otimes X(0) \oplus \Gamma_q \oplus U_q, \text{ where } U_q \in \mathbb{R}^{r \times q} \} \]

Theorem 1. Given an initial state \( X(0) \in \mathbb{R}^n_{\text{max}} \) and a state \( X \in \Omega_{q,X(0)} \) if and only if
\[ X = \Gamma_q \otimes (\Gamma_q^\perp \otimes X) + A \otimes X(0) \quad (5) \]

In which case \( \Gamma_q \otimes U_q \) is a controller drives state from \( X(0) \) to \( X = X(q) \).

Proof. If \( X \in \Omega_{q,X(0)} \), then according Definition 1, there is \( U_q \) such that the q-step state \( X = A^q \otimes X(0) \oplus \Gamma_q \oplus U_q \), is reached. Because of that \( \Gamma_q \oplus U_q \leq X \). From [2] and
[7], we get $U_q = -\Gamma_q^T \otimes' X$ is the biggest solution, then $\Gamma_q \otimes (-\Gamma_q^T \otimes' X) \leq X$. So we get

$$\Gamma_q \otimes U_q \leq \Gamma_q \otimes (-\Gamma_q^T \otimes' X) \leq X$$

(6)

With adding $A^q \otimes X(0)$ to each term in (6), we obtain:

$$A^q \otimes X(0) \oplus \Gamma_q \otimes U_q \leq A^q \otimes X(0) \oplus \Gamma_q \otimes (-\Gamma_q^T \otimes' X) \leq A^q \otimes X(0) \oplus X$$

Then we can write that $\Gamma_q \otimes (-\Gamma_q^T \otimes' X) \oplus A^q \otimes X(0) = X$, so equation (5) satisfied.

In max plus case, different from the continuo one, because the maximum operation, $\oplus$ could not be equal to the states which are less than $A^q \otimes X(0)$. In this paper, we focus the analyzing at the systems which reach a state with all of the components that greater than the final state without input. The condition of the system is called weakly reachable system.

**Definition 6.** Q-step Weakly reachable [3]. A system is said to be q-step weakly reachable, if given any $X(0)$, a controller sequence exist such that each component of the terminal state $X(q)$ can be made greater than the unforced terminal state $A^q \otimes X(0)$, there exist $U_q$ such that $(X(q))_j > (A^q \otimes X(0))_j$ for $j = 1, 2, \cdots, n$.

Before we discuss more about the weak reachability, we will give the definition as acticity first.

**Definition 7.** Asticity [3]. A $n \times m$ $G = \{g_{ij}\}$ is termed row astic if for each row $i = 1, 2, \cdots, n$, $\oplus_{j=1}^m g_{ij} \in \mathbb{R}$, Matrix $G$ is termed column astic if for each column $j = 1, 2, \cdots, m$, $\oplus_{i=1}^n g_{ij} \in \mathbb{R}$. A matrix is termed doubly astic if it in both row and column astic.

This asticity property is necessary and sufficient condition for the system to be called as weak reachability or weak observability.

**Theorem 2.** [3] A system is q-step weakly reachable if and only if $\Gamma_q$ is row astic.

**Proof.** If $\Gamma_q$ is row astic, with a great enough $U_q$, $(\Gamma_q \otimes U_q)_j > (A^q \otimes X(0))_j$ for $j = 1, 2, \cdots, n$. From the Definition 6 if a system q-step weakly reachable, then $(\Gamma_q \otimes U_q)_j > (A^q \otimes X(0))_j$, should be satisfied. So $(\Gamma_q \otimes U_q)_j$ should be finite for each $j$, because of that $\Gamma_q$ has to be row astic. Then the system is q-step weakly reachable.

Actually, row astic condition for the reachability matrix $\Gamma_q$ is needed to find that there is as least an input for each state internal transition systems. Cayley-Hamilton theorem in max plus can be used to show that if a system is not weakly reachable at q-step, then the system is also not weakly reachable at step which are more than q.

### 3. OBSERVABILITY

A system is observable if there is a final state of the system that can to determine from the measurement of the output. Because the inverse concerning the addition operator is not existing, cause the observability of the system in max plus algebra is limited. From (2) we can write a sequence q-step output as follows:
With the same recursively way, from (1) and (2) we obtain:

\[ \begin{bmatrix}
Y(0) \\
Y(1) \\
Y(2) \\
\vdots \\
Y(q-1)
\end{bmatrix} =
\begin{bmatrix}
C & e & e & e \\
CA & CB & e & e \\
CA^2 & CAB & CB & e & \cdots & e \\
\vdots & \vdots & \vdots & \vdots & \ddots & \cdots & e \\
CA^{q-1} & CA^{q-2}B & CA^{q-3} & \cdots & CAB & CB & \cdots & e
\end{bmatrix}
\begin{bmatrix}
U(0) \\
U(1) \\
U(2) \\
\vdots \\
U(q-1)
\end{bmatrix}
\]  
(7)

From (7) we can write the notation of the output and input sequence simpler, that are \( Y_q = [Y(0) \ Y(1) \ Y(2) \ \cdots \ Y(q-1)]^T \), and \( U_q = [U(0) \ U(1) \ U(2) \ \cdots \ U(q-1)]^T \).

We can also obtain q-step observability matrix, \( E_q = [C \ CA \ CA^2 \ \cdots \ CA^{q-1}] \) and matrix

\[
H_q =
\begin{bmatrix}
e & e & e & \cdots & e \\
CB & e & e & \cdots & e \\
CAB & CB & e & \cdots & e \\
\vdots & \vdots & \vdots & \ddots & \cdots & e \\
CA^{q-2}B & CA^{q-3}B & \cdots & CAB & CB
\end{bmatrix}
\]

So equation (7) can write in the different way as follows:

\[ Y(q) = E_q \otimes X(0) \oplus H_q \otimes U_q \]
(8)

With the same recursively way, from (1) and (2) we obtain:

\[
\begin{bmatrix}
Y(k) \\
Y(k+1) \\
Y(k+2) \\
\vdots \\
Y(k+q-1)
\end{bmatrix} =
\begin{bmatrix}
C & e & e & \cdots & e \\
CA & CB & e & \cdots & e \\
CA^2 & CAB & CB & \cdots & e \\
\vdots & \vdots & \vdots & \ddots & \cdots & e \\
CA^{q-1} & CA^{q-2}B & CA^{q-3} & \cdots & CAB & CB
\end{bmatrix}
\begin{bmatrix}
U(k) \\
U(k+1) \\
U(k+2) \\
\vdots \\
U(k+q-1)
\end{bmatrix}
\]  
(9)

From (9) we can write the notation of the output and input sequence simpler, that are \( Y_q = [Y(k) \ Y(k+1) \ \cdots \ Y(k+q-1)]^T \) and \( U_q = [U(k) \ U(k+1) \ \cdots \ U(k+q-1)]^T \), we can also obtain q-step observability matrix, \( E_q = [C \ CA \ CA^2 \ \cdots \ CA^{q-1}] \) and matrix

\[
H_q =
\begin{bmatrix}
e & e & e & \cdots & e \\
CB & e & e & \cdots & e \\
CAB & CB & e & \cdots & e \\
\vdots & \vdots & \vdots & \ddots & \cdots & e \\
CA^{q-2}B & CA^{q-3}B & \cdots & CAB & CB
\end{bmatrix}
\]

So equation (9) can write in the different way as follows:

\[ Y(q) = E_q \otimes X(k) \oplus H_q \otimes U_q \]
(10)

To start the discussion, we define the output of the system as the observation output that can be explained next.

**Definition 8.** [3] The observation output \( Y(q) \in R^{m \times q} \) is the output which is given by \( Y(q) = E_q \otimes X(k) \otimes H_q \otimes U_q \) with \( U_q \in \mathbb{R}_\text{max}^{p \times (q-1)} \) and \( X(0) \in \mathbb{R}_\text{max}^n \).

Gathering all of the output sequence, we will be directed to the next definition.

**Definition 9.** The set of Observable output sequence [3].

Let be given a positive integer \( p \) and \( U_q \in \mathbb{R}_\text{max}^{p \times (q-1)} \) is an input sequence, then

\[ \sum_{q \leq i} Y_q \in \mathbb{R}^{m \times q} : Y(q) = E_q \otimes X(k) \otimes H_q \otimes U_q \]
with \( X(0) \in \mathbb{R}_\text{max}^n \) is the set of observable.
Considering the necessary and sufficient condition, we can find whether an output sequence is an observable output.

**Theorem 3.** [3] Given a sequence \( Y(q) \in \mathbb{R}^{n \times q} \) and an input sequence \( U_q \in \mathbb{R}^{p \times (q-1)} \) then
\[
Y(q) \in \sum_{q \in \mathbb{Z}_{\geq 0}} \text{if and only if}
\]
\[
\mathcal{C}_q \odot (-\mathcal{C}_q^T \odot Y_q) \oplus H_q \odot U_q = Y_q
\]  
(11)

**Proof.** The proof is similar in nature and with the proof of Theorem 2.

**Definition 10.** Latest Event-Time State [3]. Given a q-length sequence of observed outputs \( Y_q \), with a sequence of inputs \( U_q \), the latest event-time state \( \gamma(k) \) which results in \( Y_q \) is
\[
\gamma(k) = \max \{ X(k) \in \mathbb{R}_{\text{max}}^n : Y_q = \mathcal{C}_q \odot X(k) \oplus H_q \odot U_q \}
\]  
(12)
where the max is over each component.

Because the latest event-time state should be infinite, then \( \gamma(k) \) define to be in \( \mathbb{R}_{\text{max}}^n \). This infinite output sequence state does not give any information about the systems state. So, we define e finite latest event-time state of the systems, which direct to the definition of weak observability.

**Definition 11.** Q-step Weakly Observable [3]. A system is q-step weakly observable if for any q-length sequence of observed outputs \( Y_q \), the latest event-time state \( \gamma(k) \) is finite and can be computed from \( Y_q \).

A necessary and sufficient condition for a system to q-step weakly observable is given next.

**Theorem 4.** [3] A system is q-step weakly observable if only if \( \mathcal{C}_q \) column astic.

**Proof.** If \( \mathcal{C}_q \) is column astic, then \( Y_q \) is finite \( \gamma(k) = -\mathcal{C}_q^T \odot Y_q \) finite too. For every \( Y_q \in \sum_{q \in \mathbb{Z}_{\geq 0}} \), with Theorem 3 we get that \( \mathcal{C}_q \odot (-\mathcal{C}_q^T \odot Y_q) \oplus H_q \odot U_q = Y_q \). Furthermore \( \gamma(k) \) is an observable output sequence in the other side, if system is q-step weakly observable. The final state should be finite and can computed from the observable output sequence \( Y_q \). By Theorem 3 we obtain \( \mathcal{C}_q \odot (-\mathcal{C}_q^T \odot Y_q) \oplus H_q \odot U_q = Y_q \). So in order \( \gamma(k) = -\mathcal{C}_q^T \odot Y_q \) to be finite, matrix \( \mathcal{C}_q \) should be column astic.

As continues variable, in max plus algebra we also have the duality of weakly reachable and weakly observable. Duality means that the property of weakly reachable can found from the property of weakly observable, vice versa.

**Example:** Given a system with matrix:

\[
A = \begin{pmatrix} \varepsilon & \varepsilon & 0 \\ \varepsilon & 2 & 0 \\ 2 & \varepsilon & \varepsilon \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} \varepsilon & 0 & \varepsilon \end{pmatrix}
\]

We will investigate the property of system, whether it’s weakly reachable or weakly observable?
1) Reachability matrix at 2-step, we obtain:

\[
\Gamma_2 = [B \ AB] = [B \ A \ \otimes \ B] = \begin{bmatrix}
0 & \max(e + 0, e + 2, 0 + e) \\
2 & \max(3 + 0, e + 2, 2 + e) \\
e & \max(e + 0, 0 + 2, e + e)
\end{bmatrix}
\]

Shown that at 2-step, the system is weakly reachable. For 3-step, we obtain the reachability matrix as:

\[
\Gamma_3 = [B \ AB \ A^2 \ B] = [B \ A \ \otimes \ B \ A^{32} \ \otimes \ B] = \begin{bmatrix}
0 & \max(e + 0, e + 2, 0 + e) \\
2 & \max(3 + 0, e + 2, 2 + e) \\
e & \max(e + 0, 0 + 2, e + e)
\end{bmatrix}
\]

For 4-step, we obtain the reachability matrix as:

\[
\Gamma_4 = [B \ AB \ A^2 \ B \ A^3 \ B] = [B \ A \ \otimes \ B \ A^{32} \ \otimes \ B \ A^{33} \ \otimes \ B] = \begin{bmatrix}
0 & \max(e + 0, e + 2, 0 + e) \\
2 & \max(3 + 0, e + 2, 2 + e) \\
e & \max(e + 0, 0 + 2, e + e)
\end{bmatrix}
\]

Shown that for 3-step, 4-step and so on the system is always weakly reachable.

2) Observability matrix at 2-step, we obtain:

\[
\mathcal{E}_2 = \begin{bmatrix}
C \\
CA
\end{bmatrix} = \begin{bmatrix}
C \\
C \otimes A
\end{bmatrix} = \begin{bmatrix}
0 & e \\
e & \max(e + e, 0 + 3, e + e)
\end{bmatrix}
\]

Shown that at 2-step, the system is weakly observable. For 3-step, we obtain the observability matrix as:

\[
\mathcal{E}_3 = \begin{bmatrix}
C \\
CA \\
CA^2
\end{bmatrix} = \begin{bmatrix}
C \\
C \otimes A \\
C \otimes A^{32}
\end{bmatrix} = \begin{bmatrix}
0 & e \\
e & \max(e + e, 0 + 3, e + e)
\end{bmatrix}
\]

For 4-step, we obtain the observability matrix as:

\[
\mathcal{E}_4 = \begin{bmatrix}
C \\
CA \\
CA^2 \\
CA^3
\end{bmatrix} = \begin{bmatrix}
C \\
C \otimes A \\
C \otimes A^{32} \\
C \otimes A^{33}
\end{bmatrix} = \begin{bmatrix}
0 & e \\
e & \max(e + e, 0 + 3, e + e)
\end{bmatrix}
\]

Shown that for 3-step, 4-step and so on the system is always weakly observable.

From 1) and 2) we get that the system had weakly reachable and weakly
observable since 2-step. Because $\Gamma_2$ is row astic and $C_2$ is column astic. Matrix $\Gamma_3$ and $\Gamma_4$ also row astic, matrix $C_3$ and $C_4$ are column astic too. We can conclude, for step-$q$ with $q \geq 2$ the system should be weakly reachable and weakly observable.

4. CONCLUSION AND FUTURE WORK

From the discussion, we obtain that the necessary and sufficient condition of weakly reachable is the row astic of its reachability matrix. Then the necessary and sufficient condition of weakly observable is the column astic of its observability matrix. If at q-step the system is weakly reachable or weakly observable, then for step-(q+1), step-(q+2), and so on the system will should be weakly reachable or weakly observable. For the future work, the discussion could be explored the strongly observable and reachable of the system. And can to determine for finite step for system is weakly reachable or weakly observable.

References